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PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR A MEDIUM WITH TWO FAMILIES OF EQUISTRESSED FIBER REINFORCEMENTS

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High values of the specific strength of fibrous composites tension-compression tests along the direction of the bonding fibers do not assure reliable operation of a structure from such material under a complex loading. Technologically, these materials possess broad possibilities for regulation of the structure and the strength of the fibers of composites is, according to the principle of their production, substantially higher than the strength of the binder. The problem of investigating the behavior of composites under the assumption of equal stress on the fibers evokes interest since the possibilities of the reinforcement are here used fully. In this paper the case is considered in which the material of the composite is either in the state of plane strain or in a state of generalized plane stress. On the basis of the relationships proposed in [1], there are equations introduced which permit finding the structure parameters of plates with an equally stressed reinforcement. Examples are examined.

1. In conformity with [1], we have for the problem under consideration in a polar coordinate system

$$\sigma_r = aE(1-\nu^2)^{-1}(\varepsilon_r + \nu\varepsilon_\vartheta) + \omega_1\sigma_1^\circ \cos^2\alpha_1 + \omega_2\sigma_2^\circ\cos^2\alpha_2 \tag{1.1}$$

$$\sigma_\vartheta = aE(1-\nu^2)^{-1}(\varepsilon_\vartheta + \nu\varepsilon_r) + \omega_1\sigma_1^\circ \sin^2\alpha_1 + \omega_2\sigma_2^\circ\sin^2\alpha_2$$

$$\tau_{r\vartheta} = aG\varepsilon_{r\vartheta} + \omega_1\sigma_1^\circ\cos\alpha_1\sin\alpha_1 + \omega_2\sigma_2^\circ\cos\alpha_2\sin\alpha_2$$

$$\varepsilon_r\cos^2\alpha_k + \varepsilon_\vartheta\sin^2\alpha_k + \varepsilon_{r\vartheta}\cos\alpha_k\sin\alpha_k = \varepsilon_k^\circ \tag{1.2}$$

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\vartheta = \frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta} + \frac{u_r}{r}, \quad \varepsilon_{r\vartheta} = \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{\partial u_\vartheta}{\partial\vartheta} - \frac{u_\vartheta}{r} \tag{1.3}$$

$$\sigma_k^\circ = E_k\varepsilon_k^\circ, \quad k = 1, 2 \tag{1.4}$$

Here σ_r, σ_ϑ, $\tau_{r\vartheta}$ are the stress components, ε_r, ε_ϑ, $\varepsilon_{r\vartheta}$ are the strain components, u_r, u_ϑ are the radial and tangential displacement components, respectively, σ_k° is the stress in the k-th family of fibers, ε_k° is the strain of the k-th family of fibers, E_k is the Young's modulus of the k-th family of fibers, E, ν, G are the elastic constants of the binder in the state of generalized plane stress of the material. If plane strain is considered, then E should be replaced by $E(1-\nu)^{-1}$, and ν by $\nu(1-\nu)^{-1}$. Henceforth, all the formulas are presented for the generalized state of stress ω_1, ω_2, $0 \leqslant \omega_1 \leqslant 1$, $0 \leqslant \omega_2 \leqslant 1$ are the intensities of the bonding of the first and second families, and α_k is the angle between the radius-vector and the k-th family of reinforcement fibers.

The system of equations (1.2) contains two unknowns u_r, u_ϑ; it is of hyperbolic type [3] with characteristics coincident with the fibers. Substituting (1.1) into the equilibrium equation and setting $k_i = \omega_i$; $\sigma_i^\circ(1-\nu^2)(aE)^{-1}$ $(i = 1, 2)$, we obtain a system of two equations in k_1, k_2

$$\cos^2\alpha_1\frac{\partial k_1}{\partial r}+r^{-1}\cos\alpha_1\sin\alpha_1\frac{\partial k_1}{\partial\vartheta}+k_1r^{-1}\cos 2\alpha_1+\cos^2\alpha_2\frac{\partial k_2}{\partial r}+ \quad (1.5)$$
$$r^{-1}\cos\alpha_2\sin\alpha_2\frac{\partial k_2}{\partial\vartheta}+k_2r^{-1}\cos 2\alpha_2=F_1(r,\vartheta)$$
$$\cos\alpha_1\sin\alpha_1\frac{\partial k_1}{\partial r}+r^{-1}\sin^2\alpha_1\frac{\partial k_1}{\partial\vartheta}+r^{-1}k_1\sin 2\alpha_1+\cos\alpha_2\sin\alpha_2\frac{\partial k_2}{\partial r}+$$
$$r^{-1}\sin^2\alpha_2\frac{\partial k_2}{\partial\vartheta}+r^{-1}k_2\sin 2\alpha_2=F_2(r,\vartheta)$$

Exactly as the system (1.2), the system (1.5) to determine the displacements is of hyperbolic type with characteristics coincident with the directions of the fibers. Let us consider the problem of determining the state of stress in a circular ring. Since the displacements and bonding intensities, and therefore the functions k_1, k_2 should be single-valued functions of the coordinates in this case, then they should be sought in the form of trigonometric series in the problem under consideration

$$u_r=u_r^\circ(r)+\sum_{n=1}^{\infty}[f_{n1}(r)\cos n\vartheta+g_{n1}(r)\sin n\vartheta] \quad (1.6)$$
$$u_\vartheta=u_\vartheta^\circ(r)+\sum_{n=1}^{\infty}[f_{n2}(r)\cos n\vartheta+g_{n2}(r)\sin n\vartheta]$$
$$k_1=k_1^\circ(r)+\sum_{n=1}^{\infty}[f_{n3}(r)\cos n\vartheta+g_{n3}(r)\sin n\vartheta] \quad (1.7)$$
$$k_2=k_2^\circ(r)+\sum_{n=1}^{\infty}[f_{n4}(r)\cos n\vartheta+g_{n4}(r)\sin n\vartheta]$$

From (1.2) we obtain a system to determine the functions u_r°, u_ϑ° and f_{ns}, g_{ns}, $s=1, 2$:

$$\frac{du_r^\circ}{dr}\cos^2\alpha_k+\frac{u_r^\circ}{r}\sin^2\alpha_k+\left(\frac{du_\vartheta^\circ}{dr}-\frac{u_v^\circ}{r}\right)\cos\alpha_k\sin\alpha_k=\varepsilon_k^\circ \quad (1.8)$$
$$\cos^2\alpha_k\frac{df_{n1}}{dr}+\frac{\sin^2\alpha_k}{r}f_{n1}+\frac{n\cos\alpha_k\sin\alpha_k}{r}g_{n1}+n\sin^2\alpha_k g_{n2}+$$
$$\left(\frac{df_{n2}}{dr}-\frac{f_{n2}}{r}\right)\cos\alpha_k\sin\alpha_k=0$$
$$\cos^2\alpha_k\frac{dg_{n1}}{dr}+\frac{\sin^2\alpha_k}{r}g_{n1}-\frac{n\sin^2\alpha_k}{r}f_{n2}-\frac{n\cos\alpha_k\sin\alpha_k}{r}f_{n1}+ \quad (1.9)$$
$$\left(\frac{dg_{n2}}{dr}-\frac{g_{n2}}{r}\right)\cos\alpha_k\sin\alpha_k=0$$

The roots of the characteristic determinant are found from the equation

$$\Delta_n(\lambda)=[\lambda^2-\lambda(1+\mathrm{tg}\,\alpha_1\,\mathrm{tg}\,\alpha_2)+(1-n^2)\,\mathrm{tg}\,\alpha_1\,\mathrm{tg}\,\alpha_2]^2+ \quad (1.10)$$
$$n^2\lambda^2(\mathrm{tg}\,\alpha_1{}^+\,\mathrm{tg}\,\alpha_2)^2=0$$

There are two pairs of complex-conjugate roots in the general case

$$\lambda_{1,2}^{(n)}=\nu_n\pm i\zeta_n,\quad \lambda_{3,4}^{(n)}=\varkappa_n\pm i_n,\quad i=\sqrt{-1} \quad (1.11)$$

where ν_n, ζ_n, $\varkappa_n$, η_n are determined from the formulas

$$\nu_n = 2^{-1}\left[(1+s) + \sqrt{\frac{r_n + x_n}{2}}\right], \quad \zeta_n = 2^{-1}\left[-n\xi + \sqrt{\frac{r_n - x_n}{2}}\right] \tag{1.12}$$

$$\varkappa_n = 2^{-1}\left[(1+s) - \sqrt{\frac{r_n + x_n}{2}}\right], \quad \eta_n = 2^{-1}\left[-n\xi + \sqrt{\frac{r_n - x_n}{2}}\right] \tag{1.13}$$

$$s = \operatorname{tg}\alpha_1 \operatorname{tg}\alpha_2, \quad \xi = \operatorname{tg}\alpha_1 + \operatorname{tg}\alpha_2, \quad x_n = (1+s)^2 - n^2\xi^2 - 4(1-n^2)s \tag{1.14}$$

$$y_n = -2(1+s)\eta\xi, \quad y_n > 0, \quad r_n = \sqrt{x_n^2 + y_n^2} \tag{1.15}$$

If the fibers of different families are orthogonal ($\operatorname{tg}\alpha_1 = -\operatorname{tg}\alpha_2$), there exists a pair of double complex-conjugate roots determinable from the equation

$$\Delta_n^*(\lambda) = [\lambda^2 - (\lambda - n^2)\operatorname{tg}^2\alpha_1]^2 = 0 \tag{1.16}$$

we have for $n = 1$

$$\Delta_n(\lambda) = [\lambda^2 - \lambda(1 + \operatorname{tg}\alpha_1 \operatorname{tg}\alpha_2)]^2 + \lambda^2(\operatorname{tg}\alpha_1 + \operatorname{tg}\alpha_2)^2 = 0 \tag{1.17}$$

and the root $\lambda = 0$ is double; the two other roots are

$$\lambda^1_{1,2} = (1 + \operatorname{tg}\alpha_1 \operatorname{tg}\alpha_2) \pm i(\operatorname{tg}\alpha_1 + \operatorname{tg}\alpha_2) \tag{1.18}$$

If $\sin\alpha_1 = 0$, we have the double root $\lambda = 0$ and two complex-conjugate roots $\lambda^{(n)}_{3,4} = 1 \pm in\operatorname{tg}\alpha_2$, but if $\cos\alpha_1 = 0$, we have a second order equation

$$\cos^2\alpha_2 \sin^2\alpha_2 (\lambda + n^2 - 1)^2 + n^2\lambda^2\cos^4\alpha_2 = 0,$$

to determine λ_2 in this case, and therefore, the characteristic equation has two complex-conjugate roots $\lambda_{1,2} = (n^2 - 1)(1 \pm n_i \operatorname{ctg}\alpha_2)^{-1}$; if $n = 1$, we have one double root $\lambda = 0$.

Therefore, in the general case ($\sin 2\alpha_1 \neq 0$) the general solution depends on four arbitrary constants, for whose determination it is sufficient to give the load on one of the contours. This is natural since in the case $\cos\alpha_1 \neq 0$ we have a Cauchy problem for a hyperbolic system of equations with data on a curve which is not a characteristic one. In the case $\cos\alpha_1 = 0$ the boundary is the characteristic and therefore, the Cauchy data depend on each other. Hence, we have "propagation" of the state of stress in this case.

The functions $f_{n1}, f_{n2}, g_{n1}, g_{n2}$ have the form

$$f_{n1}(r) = r^{\nu_n}[f_{n1}^\circ \cos(\xi_n \ln r) + f_{n2}^\circ \sin(\xi_n \ln r)] + r^{\varkappa_n}[f_{n3}^\circ \cos(\eta_n \ln r) + f_{4}^\circ{}_n \sin(\eta_n \ln r)] \tag{1.19}$$

$$g_{n1}(r) = r^{\nu_n}[-f_{n2}^\circ \cos(\xi_n \ln r) + f_{n1}^\circ \sin(\xi_n \ln r)] + r^{\varkappa_n}[-f_{n4}^\circ \cos(\eta_n \ln r) + f_{n3}^\circ \sin(\eta_n \ln r)]$$

$$f_{n2}(r) = r^{\nu_n}[f_{n3}^\circ \cos(\xi_n \ln r) + f_{n4}^\circ \sin(\xi_n \ln r) + r^{\varkappa_n}[f_{n1}^\circ \cos(\eta_n \ln r) + f_{n2}^\circ \sin(\eta_n \ln r)]$$

$$g_{n2}(r) = r^{\nu_n}[-f_{n4}^\circ \cos(\xi_n \ln r) + f_{n3}^\circ \sin(\xi_n \ln r)] + r^{\varkappa_n}[-f_{n2}^\circ \cos(\eta_n \ln r) + f_{n1}^\circ \sin(\eta_n \ln r)]$$

where f_{ni}°, $i = 1, 2, 3, 4$ are arbitrary constants. The functions f_{ni}, g_{ni}, $i = 3, 4$ are determined analogously by using the system (1.5). The functions f_{ni}, g_{ni}, $i = 1, 2$ depend on four arbitrary constants, which is the distinction between the problem under consideration and the classical plane problem of elasticity theory in which the corresponding functions f_{ni}, g_{ni} depend on eight arbitrary constants. The functions $f_{ni}, g_{ni}, i = 3, 4$ governing the bonding intensity also depend on four constants; therefore there are eight arbitrary constants for whose determination we can proceed differently (when the boundary is not a characteristic and the equations to determine the functions f_{ni}, g_{ni}, $i = 1, 2, 3, 4$ are not degenerate). These constants are determined when the load on both contours of the ring is given; u_r, u_ϑ can be fixed on one contour and k_1, k_2 on the other, etc. The appropriate trigonometric series to determine the stress and strain should converge; this constrains the possibility of variation in the boundary conditions. Let us examine particular cases.

2. Let $\alpha_1 = 0$ and $\alpha_2 = \pi / 2$. In this case the reinforcement is located along concentric circles and the radius vectors. The system (1.2) to determine the displacements u_r and u_ϑ becomes

$$\frac{\partial u_r}{\partial r} = \varepsilon_1{}^\circ, \quad \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r} = \varepsilon_2{}^\circ \tag{2.1}$$

From (2.1) we obtain

$$u_r(r, \vartheta) = \varepsilon_1{}^\circ r + a_0 + \sum_{n=1}^{\infty} [a_n \cos n\vartheta + b_n \sin n\vartheta] \tag{2.2}$$

$$u_\vartheta(r, \vartheta) = r\vartheta(\varepsilon_2{}^\circ - \varepsilon_1{}^\circ) - \frac{a_0\vartheta}{2} - \sum_{n=1}^{\infty}\left[\frac{a_n}{n}\sin\vartheta - \frac{b_n}{n}\cos n\vartheta\right] \tag{2.3}$$

Since the tangential displacement u_ϑ should be periodic in ϑ (a closed ring is considered), it follows from (2.3) that $a_0 \equiv 0$, $\varepsilon_2{}^0 \equiv \varepsilon_1{}^0$. It follows from (2.2) and (2.3) that it is sufficient to give one of the displacements u_r or u_ϑ on either of the ring contours in order to determine the unknown coefficients a_n and b_n in (2.2), (2.3). Let us recall that both the displacements u_r and u_ϑ are given independently on the boundary in the classical theory of elasticity. The ambiguity originating is associated with the fact that a characteristic Cauchy problem for a hyperbolic system of equations is solved.

If we have a system of boundary conditions in stresses

$$\sigma_r = a\sigma_r{}^\circ + k_1 |_{r=R} = p_1(\vartheta), \quad \tau_{r\vartheta} = \frac{1-\nu}{2}\varepsilon_{r\vartheta}|_{r=R} = p_2(\vartheta) \tag{2.4}$$

then since the stress $\sigma_r{}^\circ$ in the binder is constant, the coefficients a_n and b_n are determined when giving the tangential load along the contour and are expressed by the formulas $a_n = Rp_{n1}(1-n^2)^{-1}$, and $b_n = -R(1-n^2)^{-1}p_{n2}$, where p_{n1} and p_{n2} are, respectively, the coefficients of $\cos n\vartheta$ and $\sin n\vartheta$ in the expansion of $p_2(\vartheta)$ in trigonometric series, and $\varepsilon_{r\vartheta}$ has the form

$$\varepsilon_{r\vartheta} = Rr^{-1}g(\vartheta) = (Rr^{-1})\sum_{n=1}^{\infty}[p_{n1}\cos n\vartheta - p_{n2}\sin n\vartheta]$$

The functions k_1 and k_2 are determined from the equations

$$\frac{\partial k_1}{\partial r} + \frac{k_1}{r} - \frac{k_2}{r} = \frac{(1-\nu)}{2}\frac{R}{r^2}g'(\vartheta), \quad k_2 = -\frac{(1-\nu)R}{2r}g(\vartheta) \tag{2.5}$$

From this last equation it follows

$$\omega_2(r, \vartheta) = -aG\sigma_0{}^{-1}Rr^{-1}g(\vartheta) \tag{2.6}$$

Since there should always be $0 \leqslant \omega_2 \leqslant 1$, it follows from (2.6) that $g(\vartheta) \leqslant 0$ for $\sigma_0 > 0$ (i.e., when the fiber along the concentric circles are stretched), and conversely,

$g(\vartheta) \geqslant 0$ for $\sigma_0 < 0$. Moreover, ω_2 should satisfy the inequality $\omega_2 \leqslant 1$, which results in a constraint on the load $p_2(\vartheta)$ in conformity with (2.4) and (2.6). An analogous inequality should be satisfied for k_1. This constraint on k_1 results in a corresponding constraint on the load $p_1(\vartheta)$.

If a solid disc is considered, then as follows from (2.5), $k_1 \equiv k_2 \equiv 0$, $\varepsilon_{r\vartheta} = 0$ and the plate meterial will be in a state of homogeneous strain. No equally stressed bonded structure in the form of a solid disc exists with reinforcement along the radius vectors and concentric circles.

3. Let us examine the case when $\alpha_1 = 0$ and $0 < \alpha_2 = \beta < \pi / 2$. In this case one of the families of the fibers is located along the radius-vectors, and the other lies arbitrarily without, however, being coincident with concentric circles. The system (1.2) has the form $\partial u_r / \partial r = \varepsilon_1^\circ$

$$\frac{\partial u_r}{\partial r}\cos^2\beta + \left(\frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r}\right)\sin^2\beta + \left(\frac{1}{r}\frac{\partial u_r}{\partial\vartheta} + \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r}\right)\cos\beta\sin\beta = \varepsilon_2^\circ \tag{3.1}$$

The functions $u_r^\circ(r)$ and $u_\vartheta^\circ(r)$ have the form

$$u_r^\circ(r) = \varepsilon_1^\circ r + C_1, \quad u_\vartheta^\circ(r) = r\left[\frac{2\ln r}{\sin 2\beta} + (\varepsilon_2^\circ - \varepsilon_1^\circ) + C_1 \operatorname{tg}\beta r^{-1} + C_2\right] \tag{3.2}$$

The parts u_r^1 and u_ϑ^1 of the functions u_r and u_ϑ, dependent on ϑ have the form

$$u_r^1 = \sum_{n=1}^{\infty}[f_{n1}\cos n\vartheta + g_{n1}\sin n\vartheta] \tag{3.3}$$

$$u_\vartheta^1 = \sum_{n=1}^{\infty}\Big[\Big\{\frac{ng_{n1} - (n^2-1)\sin\beta\cos\beta f_{n1}}{n^2\sin^2\beta + \cos^2\beta} + r\,[u_{n3}\cos(n\operatorname{tg}\beta\ln r) + u_{n4}\sin(n\operatorname{tg}\beta\ln r)]\Big\}\cos n\vartheta + \Big\{-\frac{nf_{n1} + (n^2-1)\sin\beta\cos\beta g_{n1}}{n^2\sin^2\beta + \cos^2\beta} + r\,[-u_{n4}\cos(n\operatorname{tg}\beta\ln r) + u_{n3}\sin(n\operatorname{tg}\beta\ln r)]\Big\}\sin n\vartheta\Big]$$

If the material strain occurs without rotation of a small element (the elementary rotation of an element of medium in polar coordinates $2\chi = -\partial u_r/\partial\vartheta + \partial u_\vartheta/\partial r + u_\vartheta/r$ equals zero), we obtain $u_\vartheta' \equiv 0$, $u_r' \equiv 0$, $u_r^\circ = \varepsilon_1^\circ r$, $u_\vartheta^\circ = 0$, $\varepsilon_2^\circ \equiv \varepsilon_1^\circ$; therefore, the material is hence in a state of homogeneous strain. Then the functions k_1 and k_2 become: $k_2 = D_0 r^{-2}$, $k_1 = -D_0 r^{-2} + D_1 r^{-1}$ and are determined uniquely when the load is given on one of the contours. In the case of a solid disc with a self-equilibrated load k_1 and k_2 should vanish because of boundedness; therefore, no equally stressed bonded structure exists in this case.

Under the assumption of independence of u_r, u_ϑ, k_1, k_2 from the polar angle, the stresses in the binder have the form

$$\begin{aligned}\sigma_r^\circ &= aE\,(1-\nu^2)^{-1}[\varepsilon_1^\circ(1+\nu) + \nu C_1 r^{-1}]\\ \sigma_\vartheta^\circ &= aE\,(1-\nu^2)^{-1}[\varepsilon_1^\circ(1+\nu) + C_1 r^{-1}],\\ \tau_{r\vartheta}^\circ &= aG\left[\frac{2}{\sin 2\beta}(\varepsilon_2^\circ - \varepsilon_1^\circ) - r^{-1}C_1\operatorname{tg}\beta - C_2\right]\end{aligned} \tag{3.4}$$

and k_1 and k_2 are hence determined by the formulas

$$\begin{aligned} k_2 &= C_1 r^{-1} \operatorname{tg} \beta - 2 (\sin 2\beta)^{-1} (\varepsilon_2^\circ - \varepsilon_1^\circ) + C_2 + D_1 r^{-2} \\ k_1 &= -C_1 r^{-1} \sin \beta \cos \beta - D_1 \cos^2\beta r^{-2} + C_1 r^{-1} \operatorname{tg} \beta \cos 2\beta + \\ &\quad D_1 r^{-2} \cos 2\beta + 2 C_1 \cos 2\beta (\sin 2\beta)^{-1} (\varepsilon_2^\circ - \varepsilon_1^\circ) \ln r - \\ &\quad C_2 \cos 2\beta \ln r + D_2 \end{aligned} \tag{3.5}$$

To determine the constants in (3.1) — (3.3) there exist different methods of giving the boundary conditions: $u_\vartheta (r, \vartheta)$ can be given on both contours of the ring and $u_r (r, \vartheta)$ can hence be determined from the solution of the problem, or u_r and u_ϑ can be given on one of the ring contours, k_2 can be given on one contour and k_1 on the other, or they can be given together on one of the ring contours, etc. Different methods can also act when specifying the load on the boundary; giving the tangential component $\tau_{r\vartheta}$ on both contours, and k_1 and k_2 can be given either together on one contour, or separately on different contours. Constraints on the possible magnitude of the boundary intensities of the bonding or the magnitude of the boundary load originate from the requirement of equilibration of the load.

In the case of a solid disc, it follows from (3.4) and (3.5) that $C_1 \equiv 0$, $D_1 \equiv 0$, $C_2 \equiv 0$ because of the boundedness of k_1 and k_2, then

$$k_1 \equiv D_2, \quad k_1 = -(\sin \beta \cos \beta)^{-1} (\varepsilon_2^\circ - \varepsilon_1^\circ); \quad \sigma_r^\circ = \sigma_\vartheta^\circ = aE\varepsilon_1^\circ (1 - \nu)^{-1}$$

$\tau_{r\vartheta}^\circ = aG (\sin \beta \cos \beta)^{-1} (\varepsilon_2^\circ - \varepsilon_1^\circ) = - aGk_1$, $0 \leqslant \omega_2 \leqslant 1$, $0 \leqslant \omega_1 \leqslant 1$, $0 \leqslant D_2 \leqslant 1$ for $\sin 2\beta > 0$, $\varepsilon_2^\circ - \varepsilon_1^\circ < 0$, for $\sin 2\beta < 0$, $\varepsilon_2^\circ - \varepsilon_1^\circ > 0$ and moreover, $| (\varepsilon_2^\circ - \varepsilon_1^\circ) (\sin \beta \cos \beta)^{-1} | \leqslant 1$. Therefore, we obtain a constraint on the fiber strain and the magnitude of the angle.

4. Let us examine the case when $\alpha_1 = \pi/2$ and $0 < \alpha_2 = \beta < \pi/2$. One of the families of reinforcement fibers is hence located along concentric circles, while the other is arbitrary without coinciding with the radius-vectors. In this case the system (1.2) is written in the form

$$\frac{\partial u_\vartheta}{\partial \vartheta} + u_r = \varepsilon_1^\circ r \tag{4.1}$$

$$r \frac{\partial u_r}{\partial r} \cos^2 \beta + \left(\frac{\partial u_\vartheta}{\partial \vartheta} + u_r \right) \sin^2 \beta + \left(\frac{\partial u_r}{\partial \vartheta} + r \frac{\partial u_\vartheta}{\partial r} - u_\vartheta \right) \cos\beta \sin\beta = \varepsilon_2^\circ r$$

The functions $u_r^\circ (r)$ and $u_\vartheta^\circ (r)$ are defined by the formulas

$$u_r^\circ = \varepsilon_1^\circ r, \quad u_\vartheta^\circ (r) = \frac{(\varepsilon_2^\circ - \varepsilon_1^\circ)}{\cos \beta \sin \beta} r \ln r + C_0 r \tag{4.2}$$

while their parts u_r^1 and u_ϑ^1, which depend on the polar angle are given by

$$u_r^1 (r, \vartheta) = \sum_{n=1}^{\infty} [n v_n (r) \cos n\vartheta - n u_n (r) \sin n\vartheta] \tag{4.3}$$

$$u_\vartheta^1 (r, \vartheta) = \sum_{n=1}^{\infty} [u_n (r) \cos n\vartheta + v_n (r) \sin n\vartheta]$$

and the functions $u_n (r)$ and $v_n (r)$ have the form

$$\begin{aligned} u_n (r) &= r^{\nu_n} [u_{n1} \cos (\xi_n \ln r) + u_{n2} \sin (\zeta_n \ln r)] \\ v_n (r) &= r^{\nu_n} [- u_{n2} \cos (\xi_n \ln r) + u_{n1} \sin (\xi_n \ln r)] \end{aligned} \tag{4.4}$$

where u_{n1}, u_{n2} are constants and ν_n and ξ_n have the form

$$\nu_n = (n^2+1)\sin^2\beta\,[\sin^2\beta + n^2\cos^2\beta]^{-1}, \qquad (4.5)$$
$$\xi_n = n(n^2+1)\cos\beta\sin\beta\,[\sin^2\beta + n^2\cos^2\beta]^{-1}$$

As is seen from (4.1) — (4.5), the number of constants is halved as compared with the preceding case. This is related to the fact that the boundary is characteristic for the system (4.1) in this case. As in the case examined in Sect. 2, it is sufficient to specify one of the displacements on any of the ring contours to determine the state of material strain, and we detect in solving the problem and specifying stresses on the boundary that it is impossible to specify the radial and tangential load components independently; in the absence of elementary rotation of the volume element, the material will be in a state of homogeneous strain.

In this case the system (1.5) is written as

$$r^{-1}\frac{\partial k_1}{\partial\vartheta} + \cos\beta\sin\beta\frac{\partial k_2}{\partial r} + r^{-1}k_2\sin 2\beta = F_2(r,\vartheta),\ k_1 = -rF_1(r,\vartheta) \qquad (4.6)$$

If the material strain is axisymmetric, we obtain

$$\varepsilon_r = \varepsilon_1^\circ,\quad \varepsilon_\vartheta = \varepsilon_1^\circ,\quad \varepsilon_{r\vartheta} = (\varepsilon_2^\circ - \varepsilon_1^\circ)(\cos\beta\sin\beta)^{-1} \qquad (4.7)$$
$$k_1 \equiv 0,\quad k_2 = -(1-\nu)(\varepsilon_2^\circ - \varepsilon_1^\circ)(\cos\beta\sin\beta)^{-2} + C_1 r^{-2}$$

In the case of a solid disc $\omega_2 = \text{const}$, and $(\varepsilon_2^\circ = -\varepsilon_1^\circ) < 0$ follows from the positivity requirement for ω_2. It also follows from (4.7) that $\omega_1 \equiv 0$ both in the case of a ring and in the case of a solid disc, i.e., in axisymmetric material strain under conditions that the fibers are equally stressed the reinforcement over the concentric circles should not be mentioned.

5. Let us consider the exceptional case: the fibers lie in orthogonal directions. Let us put $\alpha_2 = \alpha_1 + \pi/2$. In this case, the characteristic equation (1.10) to determine the functions u_r and u_ϑ becomes

$$\cos^2\alpha_1\sin^2\alpha_1\,[\lambda^2 + n^2 - 1]^2 + n^2\lambda^2\cos^2 2\alpha_1 = 0 \qquad (5.1)$$

and has four roots

$$\lambda_{1,2}^{(n)} = -ni\,|\operatorname{ctg} 2\alpha_1| \pm i\,|\sin 2\alpha_1|^{-1}\sqrt{n^2 - \sin^2 2\alpha_1} \qquad (5.2)$$
$$\lambda_{3,4}^{(n)} = ni\,|\operatorname{ctg} 2\alpha_1| \pm i\,|\sin 2\alpha_1|^{-1}\sqrt{n^2 - \sin^2 2\alpha_1}$$

in the general case $(\sin^2\alpha_1 \neq 0,\ \alpha_1 \neq \pi/4)$.

The case $\alpha_1 = \pi/4$ (reinforcement lies at an angle $\pi/4$ to the radius vector) is exceptional; in this case (5.1) has two double roots $\lambda_{1,2}^{(n)} = \pm i\sqrt{n^2-1}$. For $n=1$ we have the double root $\lambda = 0$ and two imaginary roots $\lambda_{3,4}^{(1)} = \pm 2i \operatorname{ctg} 2\alpha_1$. For $\alpha_1 = \pi/4$ and $n = 1$, Eq. (5.1) has one quadruple root $\lambda = 0$. The functions $u_r^\circ(r)$ and $u_\vartheta^\circ(r)$ have the form

$$u_r^\circ(r) = r2^{-1}(\varepsilon_1^\circ + \varepsilon_2^\circ) + C_1 r^{-1} \qquad (5.3)$$
$$u_\vartheta^\circ(r) = (\sin 2\alpha_1)^{-1}[(\varepsilon_2^\circ - \varepsilon_1^\circ)\, r\ln r - r^{-1}C_1\cos 2\alpha_1 + C_2 r]$$

The case $\alpha_1 = 0$ is not considered here since it has been examined in Sect. 2. The functions k_1 and k_2 are determined from (1.5) for $\alpha_2 = \alpha_1 + \pi/2$.

For axisymmetric strain of the material and $\alpha_1 = \pi/4$ the dimensionless stresses in the binder are determined by the two formulas

$$\sigma_{r\vartheta}{}^{\circ} = (1+\nu)(\varepsilon_1{}^{\circ}+\varepsilon_2{}^{\circ})\,2^{-1} \mp C_1(1-\nu)\,r^{-2}$$
$$\tau_{r\vartheta}{}^{\circ} = (1-\nu)(\sin 2\alpha_1)^{-1}\,[(\varepsilon_1{}^{\circ}-\varepsilon_2{}^{\circ}) + 2C_1 r^{-2}\cos 2\alpha_1]$$

and the functions k_1 and k_2 have the form

$$k_1 = 2^{-1}C_2 + D_1 r^{-2}, \quad k_2 = C_2 2^{-1} + [3C_1(1-\nu) - D_1]r^{-2}$$

Here C_1, C_2, D_1 are constants determined from the boundary conditions. To determine these constants it is sufficient to specify the load on one of the ring contours, and for example, k_2 on the other; other variations are also possible.

For any α_1 the functions f_{n1}, g_{n1}, f_{n2}, g_{n2} are determined by formulas of the type (1.19) in which $\nu_n \equiv \varkappa_n \equiv 0$.

6. The state of strain of the material was axisymmetric in all the cases considered for pure strain of the material. For $\alpha_1 \neq \pi/2$, $\alpha_2 \neq \pi/2$, $\alpha_1 \neq \alpha_2$ the displacements under axisymmetric strain are expressed by the formulas

$$u_r{}^{\circ} = r^s\,[D_1 + D\,(1-s)^{-1}r^{1-s}] \tag{6.1}$$
$$u_\vartheta{}^{\circ} = D_2 r + r\,(\cos\alpha_1 \sin\alpha_1)^{-1}\,[\varepsilon_1{}^{\circ}\ln r - D_1\,(s\cos^2\alpha_1 + \sin^2\alpha_1)\times (1-s)^{-1}r^{s-1} - D(1-s)^{-1}\ln r]$$

where

$$s = \operatorname{tg}\alpha_1 \operatorname{tg}\alpha_2 \ (\cos\alpha_1\cos\alpha_2 \neq 0, \ \sin 2\alpha_1 \neq 0) \tag{6.2}$$
$$D = [\varepsilon_1{}^{\circ}\sin 2\alpha_2 - \varepsilon_2{}^{\circ}\sin 2\alpha_1]\,[2\sin(\alpha_2-\alpha_1)\cos\alpha_1\cos\alpha_2]^{-1}$$

Therefore, $u_r{}^{\circ}$ and $u_\vartheta{}^{\circ}$ depend on two arbitrary constants D_1 and D_2 determined when the load or the displacement is specified on the ring contours. If the ring degenerates into a solid disc, then $D_1 \equiv 0$ is necessary for $\operatorname{tg}\alpha_1 \operatorname{tg}\alpha_2 < 1$ since otherwise the strains are infinite at the center of the disc. We have $s \neq 1$ in (6.1) and (6.2) since the directions of fibers of different families do not agree. Even in this case, if the material is strained without rotation of an elementary volume, we will obtain $\varepsilon_1{}^{\circ} \equiv \varepsilon_2{}^{\circ}$, $D_2 \equiv 0$. In this case $u_r{}^{\circ}$ and $u_\vartheta{}^{\circ}$ depend only on one arbitrary constant D_1.

In any case, under the assumption of pure strain of the material, its state of strain turns out to be homogeneous and the fiber of the material must be equally strained. The state of material stress hence need not be axisymmetric, and the bonding intensities are determined by the formulas presented in Sect. 5.

7. The requirements of total equilibrium, that the principal vector and principal moment of the effective forces equal zero, impose a constraint on either the selection of the reinforcement or on the selection of the limiting bonding intensities. For example for $\alpha_1 = \pi/4$, $\alpha_2 = 3\pi/4$

$$\omega_k\,|_{r=R_m} = \omega_{km}, \quad k,m = 1,2 \tag{7.1}$$

(where R_2 is the outer radius of the ring and R_1 is the inner radius) and we have from the condition that the principal moment equals zero

$$[R_2{}^2\,\omega_{22} - R_1{}^2\omega_{21}]\,[R_1{}^2\omega_1 - R_2{}^2\omega_{12}]^{-1} = E_1 E_2^{-1} \tag{7.2}$$

It follows from (7.2) that for bonding intensities given on the boundary it is impossible to select the reinforcement material arbitrarily, and on the other hand, for given Young's moduli of the reinforcement it is impossible to select the limiting bonding intensities arbitrarily. Since the binder material remains elastic for given loads by assumption, the strength condition for the binder should be satisfied. For example, if it is assumed in the case of a solid disc and homogeneous strain that the binder material is subject to the Mises yield (strength) condition with the yield point (strength) τ_*, then

$$|\varepsilon_1^\circ| \leqslant \sqrt{3}\,(1-\nu)\,(aE)^{-1}\,\tau_*$$

therefore, the admissible fiber strain depends on the strength of the binder.

Let us clarify the necessary constraints. On the contour Γ of the domain Ω let the load components be given. Then the system of equal-stress relationships and the boundary conditions are

$$\sigma_r \cos\gamma + \tau_{r\theta}\sin\gamma\,|_\Gamma = f_1(\Gamma),\quad \tau_{r\theta}\cos\gamma + \sigma_\theta\sin\gamma\,|_\Gamma = f_2(\Gamma)$$

where γ is an angle formed by the external normal of the boundary and the radius-vector,form a system of four equations in three components ε_r, ε_θ, $\varepsilon_{r\theta}$ of the strain tensor. Since the strain tensor components should be determined from the solution of the problem, then by the Kronecker-Cappelli theorem (see [2]) the rank of the matrix A of coefficients for ε_r, ε_θ, $\varepsilon_{r\theta}$ should equal the rank of the expanded matrix. Then the following equality should hold

$$-f_1B_3 + f_2B_4 - \varepsilon_1^\circ B_1 + \varepsilon_2^\circ B_2 = 0$$

where B_i, $i = 1, 2, 3, 4$ have the form

$$B_k = \frac{(aE)^2}{2(1-\nu)(1+\nu)^2}\{\nu\cos^2(\gamma-\alpha_m) - \sin^2(\gamma-\alpha_m)\}$$

$$B_{k+2} = -\frac{aE}{2(1-\nu^2)}\sin(\alpha_m-\gamma)\{\cos\alpha_k\sin(\gamma+\alpha_m) +$$
$$\cos\alpha_m\sin(\gamma+\alpha_k) + \nu[\sin\alpha_m\cos(\gamma-\alpha_k) + \sin\alpha_k\cos(\gamma-\alpha_m)]\}$$
$$k, m = 1, 2;\quad k \neq m$$

If at last one of the B_i, $i = 1, 2, 3, 4$ is different from zero, this system has a unique solution. This means that if the relationship (1. 2) is satisfied, then for known values of the bonding intensities the strain components on the boundary contour are determined uniquely by means of the known load components.

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